p-ADIC INTERPOLATION FUNCTION RELATED TO MULTIPLE GENERALIZED GENOCCHI NUMBERS

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Abstract. In the present paper, we deal with multiple generalized Genocchi numbers and polynomials. Also, we introduce analytic interpolating function for the multiple generalized Genocchi numbers attached to χ at negative integers in complex plane and we define the multiple Genocchi p-adic L-function. Finally, we derive the value of the partial derivative of our multiple p-adic l-function at $s = 0$.

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1. PRELIMINARIES

The works of generalized Bernoulli, Euler and Genocchi numbers and polynomials and their combinatorial relations have received much attention [6], [7]-[27], [30], [31], [32], [33], [34]. Generalized Bernoulli polynomials, generalized Euler polynomials and generalized Genocchi numbers and polynomials are the signs of very strong relationship between elementary number theory, complex analytic number theory, Homotopy theory (stable Homotopy groups of spheres), differential topology (differential structures on spheres), theory of modular forms (Eisenstein series), p-adic analytic numbers theory (p-adic L-functions), quantum physics(quantum Groups).

p-adic numbers also were invented by German Mathematician Kurt Hensel around the end of the nineteenth century. In spite of their being already one hundred years old, these numbers are still today enveloped in an aura of mystery within the scientific community. The p-adic integral was used in mathematical physics, for instance, the functional equation of the q-zeta function, q-stirling numbers and q-Mahler theory of integration with respect to the ring $\mathbb{Z}_p$ together with Iwasawa's p-adic L functions.

Also the p-adic interpolation functions of the Bernoulli and Euler polynomials have been treated by Tsumura [35] and Young [36]. T. Kim [7]-[23] also studied on p-adic interpolation functions of these numbers and polynomials. In [37], Carlitz originally constructed q-Bernoulli numbers and polynomials. These numbers and polynomials are studied by many authors (see cf. [8]-[29], [41]). In the last decade, a surprising number of papers appeared proposing new generalizations of the Bernoulli, Euler and Genocchi polynomials to real and complex variables.

In [7]-[24], Kim studied some families of multiple Bernoulli, Euler and Genocchi numbers and polynomials. By using the fermionic p-adic invariant integral on $\mathbb{Z}_p$, he constructed p-adic Bernoulli, p-adic Euler and p-adic Genocchi numbers and polynomials of higher order.

In this paper, by using Kim's method in [20], we derive several properties for the multiple generalized Genocchi numbers attached to χ.

As is well-known, Genocchi numbers are defined in the complex plane by the following exponential function

$$C(t)=\frac{2t}{e^{t}+1}=\sum_{n=0}^{\infty}G_{n}\frac{t^{n}}{n!},\ |t|<\pi. \tag{1.1}$$

It follows from the description that $G_0=0$, $G_1=1$, $G_2=-1$, $G_3=0$, $G_4=1$, $G_5=0,\cdots$, and $G_{2k+1}=0$ for $k=1,2,3,\cdots$.

The Genocchi polynomials are also given by the rule:

$$C(t,x)=e^{tG(x)}=\sum_{n=0}^{\infty}G_{n}(x)\frac{t^{n}}{n!}=\frac{2t}{e^{t}+1}e^{xt},$$

with the usual convention of replacing $G^{n}(x):=G_{n}(x)$ (see [8], [9] and [11]).

Let $w\in\mathbb{N}$. Then the multiple Genocchi polynomials of order w are given by [24]

$$C^{(w)}(t,x)=\left(\frac{2t}{e^{t}+1}\right)^{w}e^{xt}=\sum_{n=0}^{\infty}G_{n}^{(w)}(x)\frac{t^{n}}{n!},\ |t|<\pi. \tag{1.2}$$

Taking $x=0$ in (1.2), then we have $G_{n}^{(w)}(0):=G_{n}^{(w)}$ are called the multiple Genocchi numbers of order w.

For $f\in\mathbb{N}$ with $f\equiv 1\,(\mathrm{mod}\,2)$, we assume that χ is a primitive Dirichlet's charachter with conductor f. It is known in [28] that the Genocchi numbers associated with χ, $G_{n,\chi}$, was introduced by the following expression

$$C_{\chi}(t)=2t\sum_{\xi=1}^{f}\frac{\chi(\xi)(-1)^{\xi}e^{\xi t}}{e^{ft}+1}=\sum_{n=0}^{\infty}G_{n,\chi}\frac{t^{n}}{n!},\ |t|<\frac{\pi}{f}. \tag{1.3}$$

In this paper, we contemplate the definition of the generating function of the multiple generalized Genocchi numbers attached to χ in the complex plane. From this definition, we introduce an analytic interpolating function for the multiple generalized Genocchi numbers attached to χ. Finally, we investigate behaviour of analytic interpolating function at $s=0$.

2. ON AN ANALYTIC FUNCTION IN CONNECTION WITH THE MULTIPLE GENERALIZED GENOCCHI NUMBERS

In this part, we introduce the multiple generalized Genocci numbers attached to χ defined by

$$\begin{aligned} C_{\chi}^{(w)}(t)&=\sum_{n=0}^{\infty}G_{n,\chi}^{(w)}\frac{t^{n}}{n!} \\ &=(2t)^{w}\sum_{a_1,\cdots,a_w=1}^{f}\frac{(-1)^{a_1+\cdots+a_w}\chi(a_1+\cdots+a_w)e^{t(a_1+\cdots+a_w)}}{(e^{ft}+1)^{w}}. \end{aligned} \tag{2.1}$$

On account of (1.2) and (2.1), we easily derive the following

$$G_{n,\chi}^{(w)}=\frac{f^{n}}{f^{w}}\sum_{a_1,\cdots,a_w=1}^{f}(-1)^{a_1+\cdots+a_w}\chi(a_1+\cdots+a_w)G_{n}^{(w)}\left(\frac{a_1+\cdots+a_w}{f}\right). \tag{2.2}$$

For $s \in \mathbb{C}$, we have

$$\frac{1}{\Gamma(s)} \int_0^\infty t^{s-w-1} \left\{ (-1)^w C^{(w)}(-t,x) \right\} dt = 2^w \sum_{n_1,\cdots,n_w \geq 0} \frac{(-1)^{n_1+\cdots+n_w}}{(x+n_1+\cdots+n_w)^s}, \ x \neq 0,-1,-2,\cdots. \tag{2.3}$$

where $\Gamma(s)$ is Euler-Gamma function, which is defined by the rule

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt.$$

Thanks to (2.3), we give the multiple Genocchi-zeta function as follows: for $s \in \mathbb{C}$ and $x \neq 0,-1,-2,\cdots$,

$$\zeta_G^{(w)}(s,x) = 2^w \sum_{n_1,\cdots,n_w \geq 0} \frac{(-1)^{n_1+\cdots+n_w}}{(x+n_1+\cdots+n_w)^s}. \tag{2.4}$$

By (1.2) and (2.3), we see that

$$\zeta_G^{(w)}(-n,x) = \frac{G_{n+w}^{(w)}(x)}{\binom{n+w}{w} w!} \text{ for } n \in \mathbb{N}.$$

By utilizing from complex integral and (2.1), we obtain the following equation: for $s \in \mathbb{C}$,

$$\frac{1}{\Gamma(s)} \int_0^\infty t^{s-w-1} \left\{ (-1)^w C_\chi^{(w)}(-t) \right\} dt = 2^w \sum_{\substack{n_1,\cdots,n_w=0 \\ n_1+\cdots+n_w \neq 0}} \frac{\chi(a_1+\cdots+a_w)(-1)^{n_1+\cdots+n_w}}{(n_1+\cdots+n_w)^s}, \tag{2.5}$$

where χ is the primitive Dirichlet's character with conductor

$$f \in \mathbb{N} \text{ and } f \equiv 1 \pmod 2.$$

Because of (2.5), we give the definition Dirichlet's type of multiple Genocchi L-function in complex plane as follows:

$$L^{(w)}(s \mid \chi) = 2^w \sum_{\substack{n_1,\cdots,n_w=0 \\ n_1+\cdots+n_w \neq 0}}^\infty \frac{\chi(a_1+\cdots+a_w)(-1)^{n_1+\cdots+n_w}}{(n_1+\cdots+n_w)^s}. \tag{2.6}$$

Via the (2.1) and (2.6), we derive the following theorem:

Theorem 1. *For any $n \in \mathbb{N}$, then we have*

$$L^{(w)}(-n \mid \chi) = \frac{G_{n+w,\chi}^{(w)}(x)}{\binom{n+w}{w} w!}. \tag{2.7}$$

Let s be a complex variable, and let a and b be integer with $0 < a < F$ and $F \equiv 1 \pmod 2$.

Thus, we can consider the partial zeta function $S^{(w)}(s; a_1, \cdots, a_w \mid F)$ as follows:

$$S^{(w)}(s; a_1, \cdots, a_w \mid F) = 2^w \sum_{\substack{m_1, \cdots, m_w > 0 \\ m_i \equiv a_i (\bmod F)}} \frac{(-1)^{m_1 + \cdots + m_w}}{(m_1 + \cdots + m_w)^s} = (-1)^{a_1 + \cdots + a_w} F^{-s} \zeta_G^{(w)}\left(s, \frac{a_1 + \cdots + a_w}{F}\right). \tag{2.8}$$

Theorem 2. *The following identity holds:*

$$S^{(w)}(s; a_1, \cdots, a_w \mid F) = (-1)^{a_1 + \cdots + a_w} F^{-s} \zeta_G^{(w)}\left(s, \frac{a_1 + \cdots + a_w}{F}\right).$$

Then Dirichlet's type of multiple L-function can be expressed as the sum: for $s \in \mathbb{C}$

$$L^{(w)}(s \mid \chi) = \sum_{a_1, \cdots, a_w = 1}^{F} \chi(a_1 + \cdots + a_w) S^{(w)}(s; a_1, \cdots, a_w \mid F). \tag{2.9}$$

Substituting $s = w - n$ into (2.8), we readily derive the following: for $w, n \in \mathbb{N}$

$$\binom{n}{w} w! S^{(w)}(w - n; a_1, \cdots, a_w \mid F) = (-1)^{a_1 + \cdots + a_w} F^{n-w} G_n^{(w)}\left(\frac{a_1 + \cdots + a_w}{F}\right). \tag{2.10}$$

By (1.2), it is simple to indicate the following

$$G_n^{(w)}(x) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} G_k^{(w)} = \sum_{k=0}^{n} \binom{n}{k} x^k G_{n-k}^{(w)}. \tag{2.11}$$

Thanks to (2.8), (2.10) and (2.11), we develop the following theorem:

Theorem 3. *The following identity*

$$w! \binom{-s}{w} S^{(w)}(s + w; a_1, \cdots, a_w \mid F) = (-1)^{a_1 + \cdots + a_w} F^{-w} (a_1 + \cdots + a_w)^{-s} \sum_{k \geq 0} \binom{-s}{k} \left(\frac{F}{a_1 + \cdots + a_w}\right)^k G_k^{(w)} \tag{2.12}$$

is true.

From (2.9), (2.10) and (2.12), we have the following corollary:

Corollary 1. *The following holds true:*

$$w! \binom{-s}{w} L^{(w)}(s + w \mid \chi) = \sum_{a_1, \cdots, a_w = 1}^{F} \chi(a_1 + \cdots + a_w) (-1)^{a_1 + \cdots + a_w} F^{-w} (a_1 + \cdots + a_w)^{-s} \times \sum_{k=0}^{\infty} \binom{-s}{k} \left(\frac{F}{a_1 + \cdots + a_w}\right)^k G_k^{(w)}. \tag{2.13}$$

The values of $L^{(w)}(s \mid \chi)$ at negative integers are algebraic, hence may be regarded as lying in an extension of $\mathbb{Q}_p$. We therefore look for a p-adic function which agrees with $L^{(w)}(s \mid \chi)$ at the negative integers in the next section.

3. CONCLUSION

In this final section, we consider p-adic interpolation function of the multiple generalized Genocchi L-function, which interpolate Dirichlet's type of multiple Genocchi numbers at negative integers. Firstly, Washington constructed p-adic l-function which interpolates generalized classical Bernoulli numbers.

Here, we use some the following notations, which will be useful in remainder of paper.

Let ω denote the *Teichmüller* character by the conductor $f_\omega = p$. For an arbitrary character χ, we set $\chi_n = \chi\omega^{-n}$, $n \in \mathbb{Z}$, in the sense of product of characters. Let

$$\langle a \rangle = \omega^{-1}(a)\, a = \frac{a}{\omega(a)}.$$

Thus, we note that $\langle a \rangle \equiv 1 \,(\mathrm{mod}\, p\mathbb{Z}_p)$. Let

$$A_j(x) = \sum_{n=0}^{\infty} a_{n,j} x^n,\ a_{n,j} \in \mathbb{C}_p,\ j = 0, 1, 2, ...$$

be a sequence of power series, each convergent on a fixed subset

$$T = \left\{ s \in \mathbb{C}_p \mid |s|_p < p^{-\frac{2-p}{p-1}} \right\},$$

of $\mathbb{C}_p$ such that

(1) $a_{n,j} \to a_{n,0}$ as $j \to \infty$ for any n;

(2) for each $s \in T$ and $\epsilon > 0$, there exists an $n_0 = n_0(s, \epsilon)$ such that

$$\left| \sum_{n \geq n_0} a_{n,j} s^n \right|_p < \epsilon \text{ for } \forall j.$$

So,

$$\lim_{j \to \infty} A_j(s) = A_0(s)\,, \text{ for all } s \in T.$$

This was firstly introduced by Washington [43] to indicate that each functions $\omega^{-s}(a)\, a^s$ and

$$\sum_{k=0}^{\infty} \binom{s}{k} \left(\frac{F}{a} \right)^k B_k,$$

where F is multiple of p and f and B_k is the k-th Bernoulli numbers, is analytic on T (for more information, see [43]).

We assume that χ is a primitive Dirichlet's character with conductor $f \in \mathbb{N}$ with $f \equiv 1 \,(\mathrm{mod}\, 2)$. Then we contemplate the multiple Genocchi p-adic L-function, $L_p^{(w)}(s \mid \chi)$, which interpolates the multiple generalized Genocchi numbers attached to χ at negative integers.

For $f \in \mathbb{N}$ with $f \equiv 1 \,(\mathrm{mod}\, 2)$, let us assume that F is a positive integral multiple of p and $f = f_\chi$. We now give the definition of mutiple Genocchi p-adic L-function

as follows:

$$w!\binom{-s}{w}L_p^{(w)}\left(s+w\mid\chi\right) \tag{3.1}$$
$$=\frac{1}{F^w}\sum_{a_1,\cdots,a_w=1}^{F}\chi\left(a_1+\cdots+a_w\right)(-1)^{a_1+\cdots+a_w}\left\langle a_1+\cdots+a_w\right\rangle^{-s}$$
$$\times\sum_{k=0}^{\infty}\binom{-s}{k}\left(\frac{F}{a_1+\cdots+a_w}\right)^k G_k^{(w)}.$$

Due to (3.1), we want to note that $L_p^{(w)}\left(s+w\mid\chi\right)$ is an analytic function on $s\in T$.

For $n\in\mathbb{N}$, we have

$$G_{n,\chi_n}^{(w)}=\frac{F^n}{F^w}\sum_{a_1,\cdots,a_w=1}^{F}(-1)^{a_1+\cdots+a_w}\chi_n\left(a_1+\cdots+a_w\right)G_n^{(w)}\left(\frac{a_1+\cdots+a_w}{F}\right). \tag{3.2}$$

If $\chi_n\left(p\right)\neq0$, then $\left(p,f_{\chi_n}\right)=1$, and so the ratio $\frac{F}{p}$ is a multiple of f_{χ_n}.

Let

$$\vartheta=\left\{\frac{a_1+\cdots+a_w}{p}\mid a_1+\cdots+a_w\equiv0(\operatorname{mod}p)\text{ for some }a_i\in\mathbb{Z}\text{ with }0\leq a_i\leq F\right\}.$$

Therefore we can write the following

$$\frac{F^n}{F^w}\sum_{\substack{a_1,\cdots,a_w=1\\ p\mid a_1+\cdots+a_w}}^{F}(-1)^{a_1+\cdots+a_w}\chi_n\left(a_1+\cdots+a_w\right)G_n^{(w)}\left(\frac{a_1+\cdots+a_w}{F}\right) \tag{3.3}$$
$$=\quad p^{n-w}\frac{\left(\frac{F}{p}\right)^n}{\left(\frac{F}{p}\right)^w}\chi_n\left(p\right)\sum_{\substack{a_1,\cdots,a_w=1\\ \lambda\in\vartheta}}^{\frac{F}{p}}(-1)^{\lambda}\chi_n\left(\lambda\right)G_n^{(w)}\left(\frac{\lambda}{F/p}\right).$$

By (3.3), we define the different multiple generalized Genocchi numbers attached to χ as follows:

$$G_{n,\chi_n}^{*(w)}=\frac{\left(\frac{F}{p}\right)^n}{\left(\frac{F}{p}\right)^w}\sum_{\substack{a_1,\cdots,a_w=1\\ \lambda\in\vartheta}}^{\frac{F}{p}}(-1)^{\lambda}\chi_n\left(\lambda\right)G_n^{(w)}\left(\frac{\lambda}{F/p}\right). \tag{3.4}$$

On accounct of (3.2), (3.3) and (3.4), we attain the following

$$G_{n,\chi_n}^{(w)}-p^{n-w}\chi_n\left(p\right)G_{n,\chi_n}^{*(w)}=\frac{F^n}{F^w}\sum_{\substack{a_1,\cdots,a_w=1\\ p\nmid a_1+\cdots+a_w}}^{F}(-1)^{a_1+\cdots+a_w}\chi_n\left(a_1+\cdots+a_w\right)G_n^{(w)}\left(\frac{a_1+\cdots+a_w}{F}\right). \tag{3.5}$$

By the definition of the multiple Genocchi polynomials of order w, we write the following

$$G_n^{(w)}\left(\frac{a_1+\cdots+a_w}{F}\right)=F^{-n}\left(a_1+\cdots+a_w\right)^n\sum_{k=0}^{n}\binom{n}{k}\left(\frac{F}{a_1+\cdots+a_w}\right)^k G_k^{(w)}. \tag{3.6}$$

By (3.5) and (3.6), we have

$$G_{n,\chi_n}^{(w)} - p^{n-w}\chi_n(p)\, G_{n,\chi_n}^{*(w)} \tag{3.7}$$
$$= \frac{1}{F^w} \sum_{\substack{a_1,\cdots,a_w=1 \\ p\nmid a_1+\cdots+a_w}}^{F} (-1)^{a_1+\cdots+a_w} \chi_n(a_1+\cdots+a_w)(a_1+\cdots+a_w)^n$$
$$\times \sum_{k=0}^{n} \binom{n}{k} \left(\frac{F}{a_1+\cdots+a_w}\right)^k G_k^{(w)}$$

By (3.1) and (3.7), we readily see that

$$w!\binom{n}{w} L_p^{(w)}(w-n \mid \chi)$$
$$= \frac{1}{F^w} \sum_{\substack{a_1,\cdots,a_w=1 \\ p\nmid a_1+\cdots+a_w}}^{F} (-1)^{a_1+\cdots+a_w} \chi_n(a_1+\cdots+a_w)(a_1+\cdots+a_w)^n$$
$$\times \sum_{k=0}^{n} \binom{n}{k} \left(\frac{F}{a_1+\cdots+a_w}\right)^k G_k^{(w)} \tag{3.8}$$
$$= G_{n,\chi_n}^{(w)} - p^{n-w}\chi_n(p)\, G_{n,\chi_n}^{*(w)}$$

Consequently, we arrive at the following theorem.

Theorem 4. *The following nice identity holds true:*

$$w!\binom{-s}{w} L_p^{(w)}(s+w \mid \chi)$$
$$= \frac{1}{F^w} \sum_{a_1,\cdots,a_w=1}^{F} \chi(a_1+\cdots+a_w)(-1)^{a_1+\cdots+a_w} \langle a_1+\cdots+a_w\rangle^{-s}$$
$$\times \sum_{k=0}^{\infty} \binom{-s}{k} \left(\frac{F}{a_1+\cdots+a_w}\right)^k G_k^{(w)}.$$

Thus $L_p^{(w)}(s+w \mid \chi)$ is an analytic function on T. Additionally, for each $n \in \mathbb{N}$, we procure the following:

$$w!\binom{n}{w} L_p^{(w)}(w-n \mid \chi) = G_{n,\chi_n}^{(w)} - p^{n-w}\chi_n(p)\, G_{n,\chi_n}^{*(w)}.$$

Using Taylor expansion at $s = 0$, we have

$$\binom{-s}{k} = \frac{(-1)^k}{k} s + \cdots \ \text{ if } k \geq 1. \tag{3.9}$$

Differentiating on both sides in (3.1), with respect to s at $s = 0$, we obtain the following corollary.

Theorem 5. *Let F be a positive integral multiple of p and f. Then we have*

$$\frac{\partial}{\partial s}\left(\binom{-s}{w}L_p^{(w)}\left(s+w\mid\chi\right)\right)|_{s=0}=\frac{(-1)^w}{w}L_p^{(w)}\left(w\mid\chi\right)$$

$$=\frac{1}{w!F^w}\sum_{\substack{a_1,\cdots,a_w=1\\(a_1+\cdots+a_w,p)=1}}^{F}\chi\left(a_1+\cdots+a_w\right)(-1)^{a_1+\cdots+a_w}\left(1-\log_p\left(a_1+\cdots+a_w\right)\right)$$

$$+\frac{1}{w!F^w}\sum_{\substack{a_1,\cdots,a_w=1\\(a_1+\cdots+a_w,p)=1}}^{F}\chi\left(a_1+\cdots+a_w\right)(-1)^{a_1+\cdots+a_w}\sum_{m=1}^{\infty}\frac{(-1)^m}{m}\left(\frac{F}{a_1+\cdots+a_w}\right)^k G_k^{(w)}$$

where $\log_p x$ is the p-adic logarithm.

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